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Spatial optical similaritons in conservative nonintegrable systems

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We demonstrate analytically and numerically that stable spatial similaritons can be supported by homogeneous conservative optical media with quintic nonlinearities. Unlike previously discussed spatial similaritons, the novel waves may exist in a broad parameter regime. We also present a generic model for a quintic nonlinearity by considering a centrosymmetric nonlinear medium doped with resonant impurities in the limit of a large light carrier frequency detuning from the impurity resonance.

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Introduction. Similarity and self-similarity have been recurring themes in various branches of nonlinear physics, including nuclear physics, gas dynamics, fracture and fluid mechanics, and hydrodynamical turbulence, to mention but a few areas [1]. Lately, self-similarity has gained prominence in nonlinear optics, triggered by recent theoretical discovery [2] and experimental realization [3] of stable self-similar pulses, optical similaritons, in nonlinear fiber amplifiers in the normal dispersion regime. These advances generated a flurry of activity which is reviewed in Ref. [4]. Unlike solitons which are static (equilibrium) structures, formed as a result of the balance between diffraction or dispersion and nonlinearity, the similaritons are quintessentially nonequilibrium waves—existing in either conservative or dissipative and gain media—that maintain their structural stability (intensity profiles).

To date, research on self-similarity in optics has primarily focused on asymptotic temporal similaritons that form over long propagation distances in optical fiber amplifiers [4], although asymptotic spatial and spatio-temporal similaritons have also been studied [5,6]. At the same time, soliton-like temporal [7–9] as well as spatial $(1+1)$D [10] similaritons have been shown to exist in the media with Kerr-type nonlinearities—in the fiber or planar waveguide geometries—with gain or loss and, in general, in the presence of spatial inhomogeneities. The similaritons of a soliton-like nature can form at any propagation distances. Moreover, provided a certain compatibility condition among the parameters of the media is satisfied, they are directly related to the well-known $(1+1)$D solitons of homogeneous cubic nonlinear media; in particular, such similaritons have the same spatial or temporal profiles as the corresponding solitons. The integrability of the $(1+1)$D Kerr case guarantees stability of soliton-like similaritons. To stress a connection between the soliton-like similaritons of open inhomogeneous systems and the solitons of homogeneous integrable systems with the same nonlinearity, the term nonautonomous solitons was coined [11].

The concept of soliton-like similaritons appears, however, to extend to nonintegrable nonlinear systems. Indeed, the important role self-similarity plays in Kerr-like systems of higher dimensionality has been recently elucidated in Ref. [12]. In particular, self-focusing of $(2+2)$D beams in homogeneous Kerr media was numerically studied and a self-similar character of the beam collapse established. It was shown that regardless of the initial beam intensity profile, the central part of the beam collapses to a universal soliton-like profile, which essentially corresponds to an unstable $(2+2)$D soliton in a self-focusing Kerr medium [13,14]. All this prompts a fundamental question: Can soliton-like similaritons be supported by other than Kerr (nonintegrable) nonlinear media? And if so, under what conditions, if any, are such similaritons structurally stable?

In this Rapid Communication, we show analytically and numerically that stable $(1+1)$D spatial similaritons can propagate in media with self-focusing quintic nonlinearities. Depending on the sign of a phase chirp, novel similaritons are either self-focusing or spreading for the same—assumed to be positive hereafter—sign of the quintic medium nonlinearity. We demonstrate that soliton-like similaritons can be realized in a wide range of parameters of quintic nonlinear media. We estimate an input power required for the experimental realization of such similaritons. Similariton stability is established with the aid of numerical simulations. We also present a generic model for the quintic nonlinearity by considering a centrosymmetric medium, doped with resonant impurities whose resonant frequencies lie sufficiently far away from the beam carrier frequency. We show how the detuning from the impurity resonance can serve as a useful control parameter to engineer the desired nonlinearity of the medium.

Generic model of the quintic nonlinearity. We begin by considering a planar waveguide, filled with a centrosymmetric nonlinear medium, which is, in turn, doped with resonant impurities. The latter could be rare-earth element atoms or quantum dots (QD); eurbium- or QD-doped glasses, or semiconductors doped with quantum dots, for instance, can serve as possible realizations of the system. We treat the impurities in the two-level approximation. Within this framework, the slowly varying envelope $\mathscr{E}$ of a quasimonochromatic light beam obeys the Maxwell equation in the form

$$\frac{\partial \mathscr{E}}{\partial z} - \frac{i}{2k}\frac{\partial^2 \mathscr{E}}{\partial x^2} = \frac{iNd_{eg}\omega^2}{2\epsilon_0 kc^2}\sigma_\infty + \frac{i\omega}{c}(n_2|\mathscr{E}|^2 + n_4|\mathscr{E}|^4)\mathscr{E}. \quad (1)$$

Here the first term on the right-hand side of Eq. (1) describes the nonlinear polarization due to the resonant impurities while the remaining terms characterize the nonlinear interaction of light with the bulk medium. In Eq. (1), N is the dopant density, d_{eg} is a dipole matrix element between the excited and ground states—appropriately labeled—of an individual impurity atom; σ_∞ is a steady-state value of the atomic dipole moment, and n_2 as well as n_4 are cubic and quintic nonlinear coefficients of the bulk medium response.

The impurity dipole moment and inversion obey the standard Bloch equations [15]

$$\partial_t \sigma = -\gamma_\perp \sigma - i\Delta\sigma - i\Omega w, \tag{2}$$

and

$$\partial_t w = -\gamma_\parallel (w+1) + \frac{i}{2}(\Omega\sigma^* - \sigma\Omega^*), \tag{3}$$

where $\gamma_\perp(\gamma_\parallel)$ is a transverse (longitudinal) decay rate of the atomic dipole moment (inversion); $\Omega = 2d_{ge}\mathscr{E}/\hbar$ is the Rabi frequency, Δ is a detuning of the incident light from atomic impurity resonance, and it is assumed that in equilibrium all atoms are in a (nondegenerate) ground state. It can be shown that in the cw limit and assuming the light carrier frequency lies sufficiently far off resonance with the impurities—the detuning is much larger than the transverse relaxation rate $\Delta \gg \gamma_\perp$—we can use Eqs. (2) and (3) to adiabatically eliminate the atomic variables. The resulting steady-state dipole moment may then be developed into a series in inverse Δ as

$$\sigma_\infty \simeq \frac{2d_{ge}\mathscr{E}}{\hbar\Delta}\left(1 - \frac{4|d_{ge}|^2|\mathscr{E}|^2\gamma_\perp}{\hbar^2\Delta^2\gamma_\parallel} + \frac{16|d_{ge}|^4|\mathscr{E}|^4\gamma_\perp^2}{\hbar^4\Delta^4\gamma_\parallel^2}\right). \tag{4}$$

We can easily infer from Eqs. (1) and (4) that sufficiently far away from impurity resonances, the dopant response is approximately purely dispersive, leading to the renormalization of the nonlinearity coefficients of the bulk medium. In particular, while the first term in (4), rescaling the global phase of the field envelope, may well be omitted, the second and third ones govern the off-resonance impurity contributions to the third- and fifth-order nonlinearities, respectively. Accordingly, the analysis indicates that for a judicious choice of the frequency detuning,

$$\Delta_* = \left(\frac{4\gamma_\perp |d_{ge}|^4 N}{\gamma_\parallel n_0 \epsilon_0 \hbar^3 n_2}\right)^{1/3}, \tag{5}$$

the impurity-generated and the bulk third-order nonlinearities cancel each other, resulting in an effective renormalized quintic nonlinearity with the coefficient

$$n_{4\text{eff}} = n_4 + n_2\left(\frac{4\gamma_\perp n_2^2 n_0^2 \epsilon_0^2}{\gamma_\parallel N^2 |d_{ge}|^2}\right)^{1/3}, \tag{6}$$

where n_0 is a linear refractive index of the bulk medium, and we have used Eqs. (1), (4), and (5) to obtain Eq. (6).

Prior to introducing scaled dimensionless variables, let us estimate the order of magnitude of the necessary detuning and the effective quintic nonlinearity coefficient. To this end, we consider a realistic example of a silica-glass matrix doped with CdS QDs. In general, the transverse (phase) relaxation rate is a few times greater than the longitudinal (population) one, so we assume, for simplicity, $\gamma_\perp = 2.5\gamma_\parallel$; we also consider a typical value of the dipole matrix element to be $|d_{ge}| \simeq 10^{-28}$ Cm at a transition wavelength in the middle of the visible spectrum $\lambda \simeq 500$ nm [16]. Further, we have for silica glass $n_0 \simeq 1.45$ and $n_2 \simeq 10^{-22}$ m^2/V^2. With these numerical values, we can show that Δ_* and $n_{4\text{eff}}$ scale with the dopant density N as

$$\Delta_* \simeq 10^8 \times N^{1/3}\,\text{s}^{-1}; \quad n_{4\text{eff}} \simeq 10^{-25} \times N^{-2/3}\,\text{m}^4/\text{V}^4. \tag{7}$$

We note in passing that in deriving Eq. (7), we neglected the bulk quintic nonlinearity which is "the worst case scenario" as far as the critical power for similariton formation is concerned. Indeed, if the bulk contribution is comparable with, or greater than the impurity contribution, the critical power for similariton formation in such a material will be lower than that evaluated toward the end of this work.

It follows from Eq. (7) that the effective nonlinearity can be boosted by reducing N at the expense of decreasing the detuning. The acceptable trade off can, in fact, be accomplished for sufficiently dilute QD samples: For instance for $N \simeq 10^{12}$ m^{-3}, $n_{4\text{eff}} \simeq 10^{-33}$ m^4/V^4, while $\Delta_* \simeq 10^{12}$ s^{-1}. Note that a typical exciton lifetime of roughly 100 ps [16,17] translates into $\gamma_\perp \simeq 10^{10}$ s^{-1} such that the system is well within the confines of a purely dispersive large-detuning regime $\Delta_* \gg \gamma_\perp$.

The nonlinear wave equation for the field envelope in the medium with the renormalized nonlinearity then simplifies as

$$i\frac{\partial\mathscr{E}}{\partial z} + \frac{1}{2k}\frac{\partial^2\mathscr{E}}{\partial x^2} + \frac{kn_{4\text{eff}}}{n_0}|\mathscr{E}|^4\mathscr{E} = 0. \tag{8}$$

In the following, it will prove convenient to introduce dimensionless variables as $X = x/w_0$, $Z = z/L_D$; $L_D = kw_0^2$, being a characteristic diffraction length $U = (kn_{4\text{eff}}L_D/n_0)^{1/4}\mathscr{E}$ and recast Eq. (8) as

$$i\frac{\partial U}{\partial Z} + \frac{1}{2}\frac{\partial^2 U}{\partial X^2} + |U|^4 U = 0. \tag{9}$$

Similariton solutions and their properties. We surmise by inspection of Eq. (9) that a family of spatial similaritons is supported by the media with quintic nonlinearity; the similariton field is sought in the form

$$U(X,Z) = \frac{1}{\sqrt{W(Z)}} R\left[\frac{X - X_g(Z)}{W(Z)}\right] e^{i\Phi(X,Z)}, \tag{10}$$

where W is a similariton width and X_g is a guiding center coordinate. The self-similar profile (10) conserves the beam power $\mathscr{P}$ (per unit length in the other transverse dimension),

$$\mathscr{P} = \int dX|U|^2 = \int d\eta |U(\eta)|^2 = \text{const}, \tag{11}$$

where we introduced the similarity variable η viz.,

$$\eta = \frac{X - X_g(Z)}{W(Z)}. \tag{12}$$

Substituting the profile (10) back into Eq. (9), we obtain an ordinary differential equation for the similariton envelope whose bound solution is

$$R(\eta) = \sqrt{\sqrt{\frac{3}{8}}\text{sech}\eta}. \tag{13}$$

We note in passing that a fundamental (1 + 1)D bright soliton, supported by the quintic nonlinearity, has the same intensity profile, but it is known to be unstable [18]. The explicit dynamics of the field profile depend on the particulars of the phase evolution which is found to be given by

$$\Phi(X,Z) = -\tfrac{1}{2}C(Z)(X - X_{c0})^2 + \Theta(Z). \tag{14}$$

Here the phase chirp C obeys the equation

$$C(Z) = \frac{C_0}{1 - C_0 Z}, \tag{15}$$

where X_{c0} is the coordinate of the center of curvature, and the offset phase is given by

$$\Theta(Z) = \frac{1}{8C_0(1 - C_0 Z)}. \tag{16}$$

As a result of amplitude-phase coupling, the similariton width and guiding center dynamics are governed by the equations

$$W(Z) = 1 - C_0 Z, \tag{17}$$

and

$$X_g(Z) = X_{g0} - C_0(X_{g0} - X_{c0})Z. \tag{18}$$

It follows from Eq. (17) that depending on the chirp sign, the similaritons in quintic media can be either self-focusing—with rapidly increasing amplitude and shrinking width—or spreading at a faster rate than do freely propagating beams. As can be inferred from Eq. (18), the guiding center moves with a constant velocity $V = C_0(X_{g0} - X_{c0})$. The direction of motion depends on the sign of the chirp and relative initial positions of the guiding center and the center of curvature. Further, observe that as follows from Eqs. (10) and (17), the peak intensity of each similariton scales as

$$I_{\max}(Z) \propto \frac{1}{1 - C_0 Z}. \tag{19}$$

First, consider the self-focusing case $C_0 > 0$. In Fig. 1 we display numerical evolution of a self-focusing similariton profile on propagation in the medium. We observe that the similariton maintains its structural integrity over, at least, 80 diffraction lengths. In the inset to Fig. 1, we exhibit the evolution of the peak similariton intensity. The solid curve represents our analytical result, Eq. (19), and the crosses indicate numerically evaluated peak intensities at chosen propagation distances. After having initially increased almost linearly over small propagation distances, the peak intensity is seen to start increasing faster with the distance in accord with

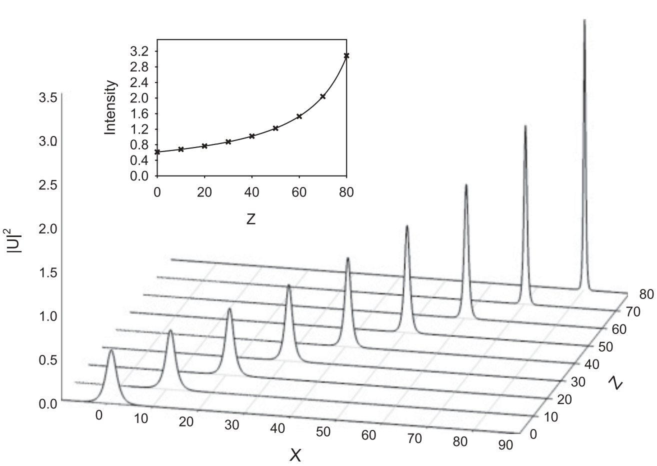

FIG. 1. Intensity profile of a self-focusing similariton as a function of the propagation distance in dimensionless variables. $X_{g0} = 1$, $X_{c0} = 100$, and $C_0 = 0.01$. Inset: the straight line represents the theoretical peak intensity as a function of the propagation distance; crosses show numerical values of the peak intensity at chosen propagation distances.

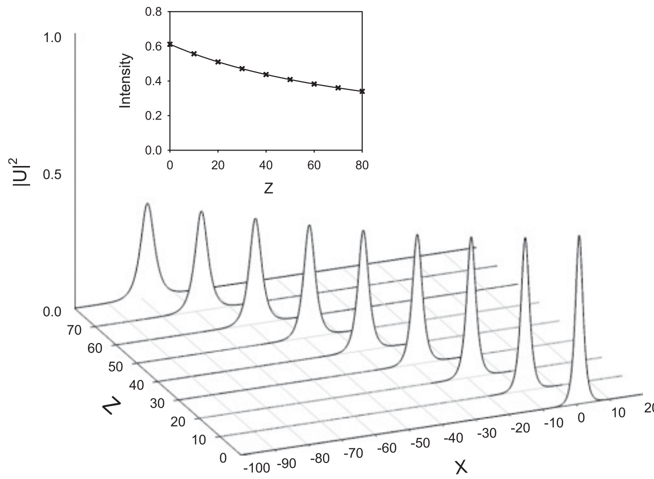

FIG. 2. Intensity profile of a self-defocusing similariton as a function of the propagation distance in dimensionless variables. $X_{g0} = 1$, $X_{c0} = 100$, and $C_0 = -0.01$. Inset: the straight line represents the theoretical peak intensity as a function of the propagation distance; crosses show numerical values of the peak intensity at chosen propagation distances.

our theory. The crosses all lie on the theoretical curve, within tiny round off numerical errors.

We can infer from Eqs. (17) and (19) that the width decreases and peak intensity increases without limit over a finite propagation distance $Z_\infty = 1/C_0$, although the total power still remains finite. Our solution becomes invalid long before the collapse takes place, though, as the paraxial approximation breaks down for small enough beam widths. We then stress that present self-focusing similaritons, just as the Townes profile for the $(2 + 1)$ D Kerr case [12], describe a self-similar stage of beam self-focusing, leading eventually to the collapse. Thus, our similaritons can be viewed as intermediate asymptotics in the spirit of Ref. [1].

Next, we consider the self-defocusing case $C_0 < 0$. The corresponding numerical evolution of the similariton profile is shown in Fig. 2. The similariton width is seen to increase with the propagation distance. Unlike the self-focusing case, there is no constraint on the range of propagation distances over which the self-defocusing solution is theoretically valid. In the inset, we again compare the theoretical behavior of the similariton peak intensity (solid line) with the numerically evaluated one (crosses). We note excellent agreement between the analytical and numerical results.

To further ascertain the structural stability of the novel self-similar solutions, we add 5% asymmetric noise to the initial similariton profile and numerically propagate the combined beam. The result is displayed in Fig. 3 and it clearly attests to the similariton stability.

Finally, we estimate the critical power needed to generate the novel similaritons. As they all have the same power by the scaling properties of Eq. (10), the latter can serve as the critical power which can be expressed as

$$P_{\rm cr} = \sqrt{\frac{3n_0}{8n_{4\rm eff}}}\left(\frac{n_0\epsilon_0 c\lambda l_\perp}{4}\right), \tag{20}$$

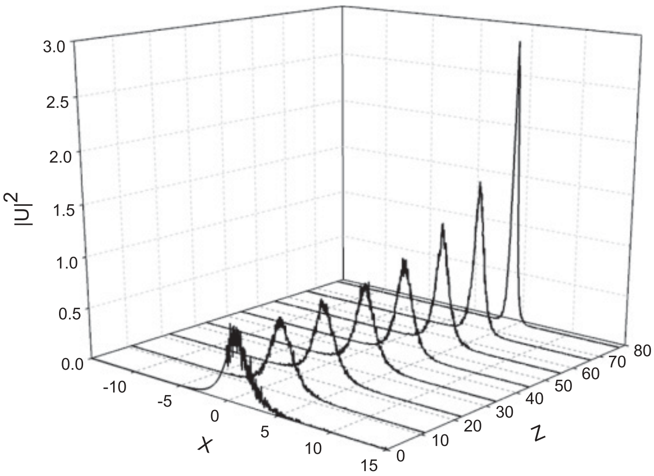

FIG. 3. Numerical evolution of the similariton with added 5% asymmetric noise. The initial parameters are $X_{g0} = 1$, $X_{c0} = 10$, and $C_0 = 0.01$. All quantities are dimensionless.

where $l_\perp$ is a waveguide width in the trapped direction. Using the previously estimated effective nonlinearity coefficient, we obtain

$$P_{\rm cr} \simeq 10^3 \times l_\perp N^{1/3} \mathrm{W}. \tag{21}$$

It is seen from Eq. (21) that for $N \simeq 10^{12}\ \mathrm{m}^{-3}$ and $l_\perp \simeq 5\ \mu\mathrm{m}$, we can arrive at the estimate, $P_{\rm cr} \simeq 50$ W. Such input powers are easily achievable with quasi-cw—millisecond long, say—laser pulses for which our cw theory is perfectly appropriate.

In summary, we demonstrated that stable spatial similaritons can be supported by quintic nonlinear media. Depending on the sign of the chirp, the similaritons can be self-focusing and self-defocusing. We also show how quintic nonlinearities can be engineered in centrosymmetric media doped with low-density impurities of resonant atoms or quantum dots. In the limit of large detuning of light from the impurity resonance frequency, the detuning serves as a convenient control parameter to design the right kind of nonlinearity, much like the phase mismatch parameter does in the case of a cascaded second harmonic generation process studied elsewhere [19].

Note added. The authors have recently become aware of Ref. [20] where self-focusing self-similar solutions of Eq. (9) were independently found by a different method.

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